Persistency of Linear Programming Formulations for the Stable Set Problem

Elisabeth Rodríguez-Heck[1], Karl Stickler[1], Matthias Walter[2], and Stefan Weltge[3]

[1]*RWTH Aachen University, Germany*
[2]*University of Twente, The Netherlands*
[3]*Technical University of Munich, Germany*

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Abstract

The Nemhauser-Trotter theorem states that the standard linear programming (LP) formulation for the stable set problem has a remarkable property, also known as (weak) *persistency*: for every optimal LP solution that assigns integer values to some variables, there exists an optimal integer solution in which these variables retain the same values. While the standard LP is defined by only non-negativity and edge constraints, a variety of stronger LP formulations have been studied and one may wonder whether any of them is persistent as well. We show that any stronger LP formulation that satisfies mild conditions cannot be persistent on all graphs, unless it is always equal to the stable-set polytope.

1 Introduction

Given an undirected graph G with node set $V(G)$ and edge set $E(G)$, and node weights $w \in \mathbb{R}^{V(G)}$, the (weighted) stable-set problem asks for finding a stable set S in G that maximizes $\sum_{v \in S} w_v$, where a set S is called stable if G contains no edge with both endpoints in S. While the stable-set problem is NP-hard, it is a common approach to maximize $w^\intercal x$ over the *edge relaxation*

$$R^{\text{edge}}_{\text{stab}}(G) := \Big\{ x \in [0,1]^{V(G)} \mid x_v + x_w \leq 1 \text{ for each edge } \{v,w\} \in E(G) \Big\}$$

and use optimal (fractional) solutions to gain insights about optimal $0/1$-solutions. Note that the $0/1$-points in the edge relaxation are precisely the characteristic vectors of stable sets in G, and that maximizing a linear objective over the edge relaxation is a linear program that can be solved efficiently. Given an optimal solution of this linear program, its objective value is clearly an upper bound on the value of any $0/1$-solution and its entries may guide initial decisions in a branch-and-bound algorithm. While this is also the case for general polyhedral relaxations, it turns out that optimal solutions of the edge relaxation have a remarkable property that allows to reduce the size of the problem by fixing some variables to provable optimal integer values.

Definition 1 (Persistency)**.** We say that a polytope $P \subseteq [0,1]^n$ has the *persistency property* if for every objective vector $c \in \mathbb{R}^n$ and every c-maximal point $x \in P$, there exists a c-maximal integer point $y \in P \cap \{0,1\}^n$ such that $x_i = y_i$ for each $i \in \{1,2,\ldots,n\}$ with $x_i \in \{0,1\}$.

Proposition 2 (Nemhauser & Trotter [9])**.** *The edge relaxation $R^{edge}_{stab}(G)$ has the persistency property for every graph G.*

In other words, the result of Nemhauser & Trotter [9] states that if $x^\star$ is an optimal solution for the edge relaxation, then there exists an optimal stable set $S^\star$ satisfying $V_1 \subseteq S^\star \subseteq V(G) \setminus V_0$, where $V_i := \{v \in V(G) \mid x^\star_v = i\}$ for $i = 0, 1$. In this case, the nodes in $V_0 \cup V_1$ can be deleted and the search only has to be performed on the remaining graph. Clearly, this reduction is significant if $x^\star$ assigns integer values to many nodes.

Hammer, Hansen and Simeone [5] provided a reduction of (Unconstrained) Quadratic Binary Programming (QBP) to the stable set problem and showed that weak persistency holds for (QBP) as well. Boros et al. [1] provided an algorithm to compute the largest possible set of variables to fix via persistencies in a quadratic binary program in polynomial time. This algorithm has been successfully used in practice to solve problems of millions of variables in the field of computer vision by reducing the problem dimension using persistencies [8, 6, 3].

In general, dual bounds obtained from the edge relaxation are quite weak, and several families of additional inequalities have been studied in order to strengthen this formulation. Examples are the clique inequalities [11], (lifted) odd-cycle inequalities [11] and clique-family inequalities [10]. Most of these families were discovered by systematically studying the facets of the *stable-set polytope* $P_{\text{stab}}(G)$, which is the convex hull of the characteristic vectors of stable sets in G. The stable-set polytope itself is known to be a complicated polytope. In particular, one cannot expect to be able to completely characterize its facial structure [7]. Thus, the following question is natural.

> *Do there exist stronger linear programming formulations for the stable set problem that also have the persistency property for every graph G?*

In this paper, we answer the question negatively. More precisely, we show that an LP formulation (satisfying mild conditions) that is stronger than the edge formulation cannot be persistent on all graphs, unless it always yields the stable set polytope.

Outline. The paper is structured as follows. We start by introducing the conditions we impose on the LP formulation in Section 2. Our main result and its consequences are presented in Section 3. Section 4 is dedicated to the proof of the main result. The paper is concluded in Section 5, where we discuss open problems.

2 LP formulations for stable set

It is clear that, for a *single* non-bipartite graph G, one can artificially construct polytopes strictly between $R^{\text{edge}}_{\text{stab}}(G)$ and $P_{\text{stab}}(G)$ that have the persistency property. For instance, if $x \in R^{\text{edge}}_{\text{stab}}(G) \setminus P_{\text{stab}}(G)$ is any point that has only fractional coordinates, then the polytope $\operatorname{conv}(P_{\text{stab}}(G) \cup x)$ has the persistency property for trivial reasons. In this work, however, we consider relaxations defined for *every* graph that arise in a more structured way.

To this end, let $\mathcal{G}$ denote the set of finite undirected simple graphs. We regard an LP *formulation* for the stable set problem as a map that assigns to every graph $G \in \mathcal{G}$ a polytope $R_{\text{stab}}(G) \supseteq P_{\text{stab}}(G)$. As an example, the edge formulation assigns $R^{\text{edge}}_{\text{stab}}(G)$ to every graph G. Next, let us specify some natural conditions that are satisfied by all prominent formulations and under which our main result holds. Each of these conditions is defined for a formulation R_{stab}.

Condition (A). The formulation R_{stab} is at least as strong as the edge formulation. Formally,

$$\text{for each } G \in \mathcal{G}, \text{ we have } P_{\text{stab}}(G) \subseteq R_{\text{stab}}(G) \subseteq R^{\text{edge}}_{\text{stab}}(G). \tag{A}$$

Condition (B). The inequalities defining R_{stab} are derived from facets of P_{stab}. Formally,

for each $G \in \mathcal{G}$, each inequality with support $U \subseteq V(G)$ that is facet-defining for $R_{\text{stab}}(G)$ is also facet-defining for $P_{\text{stab}}(G[U])$, (B)

where $G[U]$ denotes the subgraph induced by U. Note that inequalities need to define facets only on their support graph. In particular, odd-cycle inequalities satisfy (B) although in general they do not define facets [11].

Condition (C). For every graph $G \in \mathcal{G}$, validity of facet-defining inequalities of $R_{\text{stab}}(G)$ is inherited by induced subgraphs. Formally,

for each $G \in \mathcal{G}$, each inequality with support $U \subseteq V(G)$ that is facet-defining for $R_{\text{stab}}(G)$ is valid (although not necessarily facet-defining) for $R_{\text{stab}}(G[U])$. (C)

This requirement ensures that if an (irredundant) inequality arises for some graph then it must (at least implicitly) occur for all induced subgraphs for which it is defined. The reverse implication is imposed by the fourth condition, although in a more structured way. For this, we need the following definitions.

Let $G_1, G_2 \in \mathcal{G}$ and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Then the 1-*sum of* G_1 *and* G_2 *at* v_1 *and* v_2, denoted by $G_1 \oplus_{v_2}^{v_1} G_2$ is the graph obtained from the disjoint union of G_1 and G_2 by identifying v_1 with v_2. Moreover, let $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ be polytopes and let $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. The 1-*sum of* P *and* Q *at coordinates* i *and* j, denoted by $P \oplus_j^i Q$, is defined as the projection of $\operatorname{conv}(\{(x, y) \in P \times Q \mid x_i = y_j\})$ onto all variables except for y_j. Notice that this projection is an isomorphism since the variables x_i and y_j are equal.

Condition (D). For every pair of graphs $G_1, G_2 \in \mathcal{G}$, validity of inequalities is acquired by their 1-sum. Formally,

$R_{\text{stab}}(G_1 \oplus_{v_2}^{v_1} G_2) \subseteq R_{\text{stab}}(G_1) \oplus_{v_2}^{v_1} R_{\text{stab}}(G_2)$ holds for all $G_1, G_2 \in \mathcal{G}$ and all nodes $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. (D)

Also this condition is very natural since every inequality that is valid for $R_{\text{stab}}(G_1)$ is also valid for $P_{\text{stab}}(G_1 \oplus_{v_2}^{v_1} G_2)$, and hence its participation in $R_{\text{stab}}(G_1 \oplus_{v_2}^{v_1} G_2)$ is reasonable.

Before we state our main result, let us mention immediate observations, which are summarized below.

Proposition 3.

(i) R_{stab}^{edge} *and* P_{stab} *satisfy* (A)–(D).

(ii) Let R_{stab} *be any formulation satisfying* (B) *and* (C). *Then* $R_{stab}(G_1 \oplus_{v_2}^{v_1} G_2) \supseteq R_{stab}(G_1) \oplus_{v_2}^{v_1} R_{stab}(G_2)$ *holds for all* $G_1, G_2 \in \mathcal{G}$ *and all nodes* $v_1 \in V(G_1)$ *and* $v_2 \in V(G_2)$.

(iii) If R_{stab}^1 *and* R_{stab}^2 *satisfy* (A)–(D), *then* $\{R_{stab}^1(G) \cap R_{stab}^2(G)\}_{G \in \mathcal{G}}$ *also satisfies* (A)–(D).

In other words, the second observation states that the reverse inclusion of (D) is implied by (B) and (C), and the last observation is that our conditions are closed under intersection of relaxations.

Proof. It is clear that $R_{\text{stab}}^{\text{edge}}$ satisfies (A)–(D), and that P_{stab} satisfies (A) and (C). Chvátal proved Property (D) for P_{stab} by showing that the stable-set polytope of a clique-sum of two graphs is obtained from the stable-set polytopes of these two graphs without adding inequalities (see Theorem 4.1 in [2]). To see Property (B) for P_{stab}, observe that the stable-set polytope of an induced subgraph is isomorphic to a face defined by the nonnegativity constraints of the removed nodes. This shows (i).

To see (ii), let $G = G_1 \oplus_{v_2}^{v_1} G_2$, and consider an inequality that is facet-defining for $R_{\text{stab}}(G)$ and has support on $U \subseteq V(G)$. By (B), it is facet-defining for $P_{\text{stab}}(G[U])$. By (i), $P_{\text{stab}}(G[U])$ satisfies (D), that is, the support U has to satisfy $U \subseteq V(G_1)$ or $U \subseteq V(G_2)$. By (C), the inequality must be valid for $R_{\text{stab}}(G_1)$ or $R_{\text{stab}}(G_2)$, which concludes the proof.

For (iii), Property (A) is trivially satisfied, while the other three properties can be shown by inspection of individual inequalities of R^1_{stab} and R^2_{stab}. □

3 Results

We say that two formulations R^1_{stab} and R^2_{stab} are *equivalent* if $R^1_{\text{stab}}(G) = R^2_{\text{stab}}(G)$ holds for every $G \in \mathcal{G}$, in which case we write $R^1_{\text{stab}} \equiv R^2_{\text{stab}}$. We can now state our main result.

Theorem 4. *Let R_{stab} be a formulation satisfying (A)–(D). Then $R_{stab}(G)$ has the persistency property for all graphs $G \in \mathcal{G}$ if and only if $R_{stab} \equiv R^{edge}_{stab}$ or $R_{stab} \equiv P_{stab}$.*

Sufficiency follows from Proposition 2 and from the fact that P_{stab} is an integral polytope. Before we prove necessity in Section 4, let us mention some direct implications of Theorem 4 for known relaxations.

Corollary 5. *The clique relaxation*

$$R^{clq}_{stab}(G) = \left\{ x \in \mathbb{R}^{V(G)} \mid x(V(C)) \leq 1 \text{ for each clique } C \text{ of } G \right\}$$

does not have the persistency property for all graphs $G \in \mathcal{G}$.

Proof. It is easy to see that $R^{\text{clq}}_{\text{stab}}$ satisfies Properties (A) and (D). For Properties (B) and (C), consider a clique C of some graph $G \in \mathcal{G}$. Clearly, C is also a clique of $G[V(C)]$ and the inequality is known to be facet-defining for $P_{\text{stab}}(G[V(C)])$ (see Theorem 2.4 in [11]). □

Also the relaxation based on odd-cycle inequalities satisfies these properties, although the inequalities are generally not facet-defining.

Corollary 6. *The odd-cycle relaxation*

$$R^{oc}_{stab}(G) = \left\{ x \in R^{edge}_{stab}(G) \mid x(V(C)) \leq \frac{|V(C)| - 1}{2} \text{ for each chordless odd cycle } C \text{ of } G \right\}$$

does not have the persistency property for all graphs $G \in \mathcal{G}$.

Proof. It is easy to see that $R^{\text{oc}}_{\text{stab}}$ satisfies Properties (A) and (D). For Properties (B) and (C), consider a chordless odd cycle C of some graph $G \in \mathcal{G}$. Clearly, C is also a chordless odd cycle of $G[V(C)]$, and the odd-cycle inequality is facet-defining for $P_{\text{stab}}(G[V(C)])$ (see Theorem 3.3 in [11]). □

Using Proposition 3 (iii), we obtain the same result for their intersection.

Corollary 7. *The intersection of the clique and the odd-cycle relaxations*

$$R_{stab}(G) = R^{clq}_{stab}(G) \cap R^{oc}_{stab}(G)$$

does not have the persistency property for all graphs $G \in \mathcal{G}$.

Strong persistency. Hammer, Hansen and Simeone [4] considered a variant of the persistency property that considers coordinates that are fixed to the same integer for all optimal solutions. For every graph G and objective vector $c \in \mathbb{R}^{V(G)}$ they showed the following. If there is a node $i \in V(G)$ together with a value $b \in \{0, 1\}$ for which every c-maximal solution $x \in R^{\text{edge}}_{\text{stab}}(G)$ satisfies $x_i = b$, then also every c-maximal solution $y \in P_{\text{stab}} \cap \{0, 1\}^{V(G)}$ satisfies $y_i = b$. In the pseudo-Boolean optimization literature this is also referred as the *strong persistency* property of the edge relaxation. The necessity proof of Theorem 4 will show that our main result also holds for this notion of persistency.

Vertex cover. A vertex cover in a graph $G = (V, E)$ is a set $C \subseteq V$ such that C contains at least one endnode of each edge of G. Clearly, C is a vertex cover in G if and only if $V \setminus C$ is a stable set in G. One consequence of this observation is that the map $\pi : V \to V$ defined via $\pi(x)_v = 1 - x_v$ for each $v \in V$ maps $P_{\text{stab}}(G)$ to the *vertex cover polytope*, which is defined as the convex hull of characteristic vectors of vertex covers in G. Similarly, a natural linear programming relaxation for vertex cover is $\pi(R^{\text{edge}}_{\text{stab}}(G))$, which can for instance be strengthened by inequalities that correspond to cliques or odd cycles. Since also persistency is maintained under the map π, all our results also hold for vertex cover.

4 Proof of the main result

Let us fix any formulation R_{stab} over $\mathcal{G}$ satisfying Properties (A)–(D). To prove the "only if" implication of Theorem 4 we have to verify that if $R_{\text{stab}} \not\equiv R^{\text{edge}}_{\text{stab}}$ and $R_{\text{stab}} \not\equiv P_{\text{stab}}$, then $R_{\text{stab}}(G)$ does not have the persistency property for all graphs $G \in \mathcal{G}$. Equivalently, we have to prove the following:

> If there exist graphs $G_1, G_2 \in \mathcal{G}$ with $R_{\text{stab}}(G_1) \neq R^{\text{edge}}_{\text{stab}}(G_1)$ and $R_{\text{stab}}(G_2) \neq P_{\text{stab}}(G_2)$, then there exists a graph $G^\star$ for which the polytope $R_{\text{stab}}(G^\star)$ does not have the persistency property. (♢)

Given G_1 and G_2, we will provide an explicit construction of $G^\star$ and show that $R_{\text{stab}}(G^\star)$ does not have the persistency property. To see the latter, we will give an objective vector $c^\star \in \mathbb{R}^{V(G^\star)}$ such that every $c^\star$-maximal solution over $R_{\text{stab}}(G^\star)$ has a certain coordinate equal to zero while every $c^\star$-maximal stable set in $G^\star$ contains the corresponding node.

The graph $G^\star$ will consist of an "inner" graph G^{in} with $R_{\text{stab}}(G^{\text{in}}) \neq R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$ and $|V(G^{\text{in}})| - 1$ copies of an "outer" graph G^{out} with $R_{\text{stab}}(G^{\text{out}}) \neq P_{\text{stab}}(G^{\text{out}})$. Each copy of G^{out} is attached to a vertex of G^{in} via the 1-sum operation. Note that such graphs $G^{\text{in}}, G^{\text{out}}$ exist due to the hypothesis of (♢). Among all such graphs, we will make particular choices satisfying some additional properties that we specify in the next sections.

We will illustrate our definitions and the steps of the proof by providing two running examples.

Example 1. Consider the formulation $R^{\text{oc5}}_{\text{stab}}$ defined via

$$R^{\text{oc5}}_{\text{stab}}(G) = \Big\{x \in R^{\text{edge}}_{\text{stab}}(G) \mid x(V(C)) \leq \frac{|V(C)| - 1}{2} \quad \text{for each chordless odd cycle } C \text{ of } G \text{ with at least } 5 \text{ nodes} \Big\}.$$

The hypothesis of (♢) is satisfied for $R^{\text{oc5}}_{\text{stab}}$ because the odd cycle C_5 is such that $R^{\text{oc}}_{\text{stab}}(C_5) \neq R^{\text{edge}}_{\text{stab}}(C_5)$ and the complete graph K_3 is such that $R^{\text{oc5}}_{\text{stab}}(K_3) \neq P_{\text{stab}}(K_3)$.

Example 2. Consider the odd-cycle formulation $R^{\text{oc}}_{\text{stab}}$. The hypothesis of (♢) is satisfied for $R^{\text{oc}}_{\text{stab}}$ because the odd cycle C_3 is such that $R^{\text{oc}}_{\text{stab}}(C_3) \neq R^{\text{edge}}_{\text{stab}}(C_3)$ and the complete graph K_4 is such that $R^{\text{oc}}_{\text{stab}}(K_4) \neq P_{\text{stab}}(K_4)$.

4.1 The graph G^{out}

In the definition of the auxiliary graph G^{out} we will make use of the following lemma. In what follows, for a polytope $P \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{R}^n$, let us denote the optimal face of P induced by c by $\text{opt}(P, c) := \arg\max\{c^\intercal x \mid x \in P\}$.

Lemma 8. *Let $P, Q \subseteq \mathbb{R}^n$ be polytopes. If there exists a vector $c \in \mathbb{R}^n$ such that $\dim(\text{opt}(Q, c)) < \dim(\text{opt}(P, c))$, then there exists a vector $c' \in \mathbb{R}^n$ such that $\text{opt}(Q, c')$ is a vertex of Q, while $\text{opt}(P, c')$ is not a vertex of P.*

Proof. See Appendix A. □

The graph G^{out} is now defined through the following statement.

Claim 9. *There exists a graph $G^{out} \in \mathcal{G}$, a vector $c^{out} \in \mathbb{R}^{V(G^{out})}$ and a node $v^{out} \in V(G^{out})$ such that $\operatorname{opt}(R_{stab}(G^{out}), c^{out}) = \{\hat{x}\}$ holds with $\hat{x}_{v^{out}} \geq \frac{1}{2}$ and such that $\operatorname{opt}(P_{stab}(G), c^{out})$ contains a vertex $\bar{x} \in \{0,1\}^{V(G^{out})}$ with $\bar{x}_{v^{out}} = 0$.*

Proof. Let $G \in \mathcal{G}$ be such that $R_{\text{stab}}(G) \neq P_{\text{stab}}(G)$. Such a graph exists by hypothesis of ($\diamondsuit$). By Property (A), there exists an inequality $a^\intercal x \leq \delta$ that is facet-defining for $P_{\text{stab}}(G)$, but not valid for $R_{\text{stab}}(G)$.

We claim that the face $\operatorname{opt}(R_{\text{stab}}(G), a)$ is not a facet of $R_{\text{stab}}(G)$. Assume for a contradiction that $\operatorname{opt}(R_{\text{stab}}(G), a)$ is a facet of $R_{\text{stab}}(G)$ and define $\delta' := \max\{a^\intercal x \mid x \in R_{\text{stab}}(G)\}$. Since $a^\intercal x \leq \delta$ is not valid for $R_{\text{stab}}(G)$, we have $\delta' > \delta$. Property (B) implies that $a^\intercal x \leq \delta'$ is facet-defining for $P_{\text{stab}}(G[\operatorname{supp}(a)])$, and in particular, equality holds for the characteristic vector of some stable set $S \subseteq V(G[\operatorname{supp}(a)])$. Since S is also a stable set in G, this contradicts the assumption that $a^\intercal x \leq \delta$ is valid for $P_{\text{stab}}(G)$.

By Lemma 8, there exists a vector $c \in \mathbb{R}^n$ such that $\operatorname{opt}(R_{\text{stab}}(G), c) = \{\hat{x}\}$ and $\operatorname{opt}(P_{\text{stab}}(G), c)$ has (at least) two vertices $\bar{x}^1, \bar{x}^2 \in \{0,1\}^{V(G)}$. Since $\bar{x}^1 \neq \bar{x}^2$, there exists a coordinate $u \in V(G)$ at which they differ and we can assume $\bar{x}^1_u = 0$ and $\bar{x}^2_u = 1$ without loss of generality. If $\hat{x}_u \geq \frac{1}{2}$, we can choose $G^{\text{out}} := G$, $c^{\text{out}} := c$ and $v^{\text{out}} := u$. Together with $\hat{x}$ and $\bar{x}^1$, they satisfy the requirements of the lemma.

Otherwise, let G' be the graph G with an additional edge $\{u, u'\}$ attached at u. Formally, let G'' be the graph consisting of a single edge $\{u, u'\}$ and let $G' := G \oplus^u_u G''$. By Property (D), $R_{\text{stab}}(G') = R_{\text{stab}}(G) \oplus^u_u R_{\text{stab}}(G'')$ holds. Since G'' is a single edge, $R^{\text{edge}}_{\text{stab}}(G'') = P_{\text{stab}}(G'')$ holds. Thus, $R_{\text{stab}}(G')$ is described by all inequalities that are valid for $R_{\text{stab}}(G)$ together with $x_{u'} \geq 0$ and $x_u + x_{u'} \leq 1$. Hence, for a sufficiently small $\varepsilon > 0$ and the objective vector $c' \in \mathbb{R}^{V(G')}$ with $c'_{u'} = \varepsilon$, $c'_u = c_u + 2\varepsilon$ and $c'_v = c_v$ for all $v \in V(G) \setminus \{u\}$, the maximization of c' over $R_{\text{stab}}(G')$ yields a unique optimum $\hat{x}' \in \mathbb{R}^{V(G')}$ with $\hat{x}'_v = \hat{x}_v$ for all $v \in V(G)$ and $\hat{x}'_{u'} = 1 - \hat{x}'_u > \frac{1}{2}$, while the maximization of c' over $P_{\text{stab}}(G')$ admits an optimum $\bar{x}' \in \mathbb{R}^{V(G')}$ with $\bar{x}'_u = 1$ and $\bar{x}'_{u'} = 0$. Now, $G^{\text{out}} := G'$, $c^{\text{out}} := c'$ and $v^{\text{out}} := u'$ together with $\hat{x}'$ and $\bar{x}'$ satisfy the requirements of the lemma. □

Example 1 (Continued). For $R^{\text{oc5}}_{\text{stab}}$, choose G in the proof of Claim 9 to be K_3, the complete graph on nodes $\{A, B, C\}$. We assume that vectors in $\mathbb{R}^{V(G)}$ are indexed in the order A, B, C. The clique inequality $x_A + x_B + x_C \leq 1$ is facet-defining for $P_{\text{stab}}(G)$ but not valid for $R^{\text{oc5}}_{\text{stab}}(G)$. Moreover, for $c = (1,1,1)^\intercal$, $\operatorname{opt}(R^{\text{oc}}_{\text{stab}}(G), c) = \{\hat{x}\}$ with $\hat{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\intercal$ while $\operatorname{opt}(P_{\text{stab}}(G), c)$ contains the three stable sets defined by selecting a single node in G, and hence has dimension 2. Consider $\bar{x}_1 = (0,1,0)^\intercal$ and $\bar{x}_2 = (1,0,0)^\intercal$ and choose $u = A$. Following the proof of Claim 9, we obtain $G^{\text{out}} = G$, $v^{\text{out}} = A$ and $c^{\text{out}} = c$ as depicted in Figure 1a.

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, choose G in the proof of Claim 9 to be K_4, the complete graph on nodes $\{A, B, C, D\}$. We assume that vectors in $\mathbb{R}^{V(G)}$ are indexed in the order A, B, C, D. The clique inequality $x_A + x_B + x_C + x_D \leq 1$ is facet-defining for $P_{\text{stab}}(G)$ but not valid for $R^{\text{oc}}_{\text{stab}}(G)$. Moreover, for $c = (1,1,1,1)^\intercal$, $\operatorname{opt}(R^{\text{oc}}_{\text{stab}}(G), c) = \{\hat{x}\}$ with $\hat{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\intercal$ while $\operatorname{opt}(P_{\text{stab}}(G), c)$ contains the four stable sets defined by selecting a single node in G, and hence has dimension 3. Consider $\bar{x}_1 = (0,1,0,0)^\intercal$ and $\bar{x}_2 = (1,0,0,0)^\intercal$ and choose $u = A$. Since $\hat{x}_u < \frac{1}{2}$, we introduce an additional edge $\{u, u'\} = \{A, A'\}$ and the graph G'' consisting of these two nodes and the single edge. Following the proof of Claim 9, $G^{\text{out}} = G \oplus^A_A G''$ and $v^{\text{out}} = A'$. Finally, c^{out} can be defined by setting $\varepsilon = \frac{1}{3}$. Figure 1b illustrates G^{out} and c^{out}. The unique optimum of maximizing c^{out} over $R^{\text{oc}}_{\text{stab}}(G^{\text{out}})$ is $\hat{x}_v = \frac{1}{3}$ for $v \in \{A, B, C, D\}$ and $\hat{x}_{A'} = \frac{2}{3}$, while selecting only node A is an optimal stable set for c^{out}.

4.2 The graph G^{in}

Among all graphs $G \in \mathcal{G}$ with $R_{\text{stab}}(G) \neq R^{\text{edge}}_{\text{stab}}(G)$ we choose G^{in} to have a minimum number of nodes. Note that G^{in} exists by hypothesis of ($\diamondsuit$). We assume $V(G^{\text{in}}) = \{1, 2, \ldots, n\}$. Let $Ax \leq b$ (with

(a) G^{out} and c^{out} for Example 1.

(b) G^{out} and c^{out} for Example 2.

Figure 1: Illustrations of graph G^{out} and objective vector c^{out} for the two running examples.

$A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$) be the system containing inequalities for all facets of $R_{\text{stab}}(G^{\text{in}})$ that are not valid for $R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$. Note that $m \geq 1$ and $n \geq 3$ hold by assumption on G^{in}.

Claim 10. *$A_{i,j} \geq 1$ holds for every $i \in \{1, 2, \ldots, m\}$ and every $j \in \{1, 2, \ldots, n\}$.*

Proof. It is a basic fact that every facet-defining inequality of a stable-set polytope that is not a nonnegativity constraint is of the form $a^\intercal x \leq \beta$ for some nonnegative vector $a \in \mathbb{R}^n$ (see Section 9.3 in [12]). Assume, $A_{i,j} = 0$ holds for some indices i, j. By Property (C), $A_{i,\star} x \leq b_i$ is valid for $R_{\text{stab}}(G[\text{supp}(A_{i,\star})])$, while it is not valid for $R^{\text{edge}}_{\text{stab}}(G^{\text{in}}[\text{supp}(A_{i,\star})])$. This contradicts the minimality assumption for G^{in}. □

For both our examples, Claim 10 is easy to verify.

Example 1 (Continued). For $R^{\text{oc5}}_{\text{stab}}$, choose $G^{\text{in}} = C_5$ to be a cycle on nodes $\{1, 2, 3, 4, 5\}$. The system $Ax \leq b$ consists of just the odd-cycle inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$.

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, choose $G^{\text{in}} = C_3$ to be a triangle on nodes $\{1, 2, 3\}$. The system $Ax \leq b$ consists of just the triangle inequality $x_1 + x_2 + x_3 \leq 1$.

4.3 The graph $G^\star$

For each $j \in \{2, 3, \ldots, n\}$ let G^j be an isomorphic copy of G^{out} such that $V(G^j) \cap V(G^k) = \varnothing$ whenever $j \neq k$. Let $c^j \in \mathbb{R}^{V(G^j)}$ and $v^j \in V(G^j)$ be the vector and node corresponding to c^{out} and v^{out} in Claim 9, respectively. Now $G^\star$ is defined as the 1-sum of G^{in} with all G^j at the respective nodes $j \in V(G^{\text{in}})$ and $v^j \in V(G^j)$, i.e., $G^\star := G^{\text{in}} \oplus^2_{v^2} G^2 \oplus^3_{v^3} \cdots \oplus^n_{v^n} G^n$, where the $\oplus$-operator has to be applied from left to right. Note that we have

$$R_{\text{stab}}(G^\star) = R_{\text{stab}}(G^{\text{in}}) \oplus^2_{v^2} R_{\text{stab}}(G^2) \oplus^3_{v^3} \cdots \oplus^n_{v^n} R_{\text{stab}}(G^n)$$

by Property (D).

Example 1 (Continued). For $R^{\text{oc5}}_{\text{stab}}$, $G^\star$ consists of $G^{\text{in}} = C_5$ and $G^j = K^j_3$ for $j \in \{2, 3, 4, 5\}$ as depicted in Figure 2a.

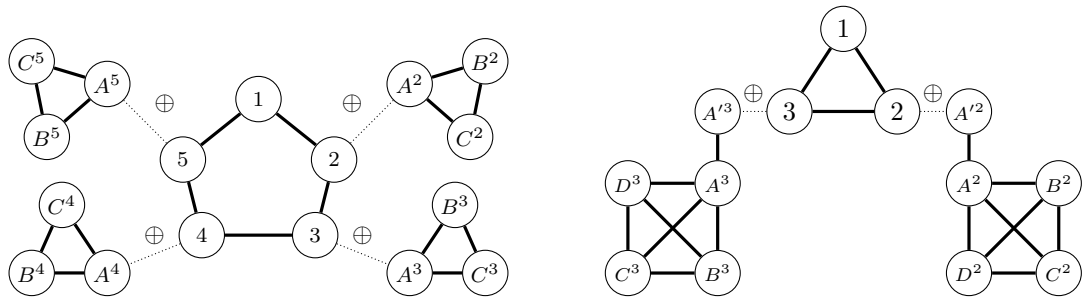
Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, $G^\star$ consists of $G^{\text{in}} = C_3$ and $G^j = K^j_4 \oplus^{A^j}_{A^j} K^j_2$ for $j \in \{2, 3\}$ as depicted in Figure 2b.

4.4 The objective vector

It remains to construct an objective vector $c^\star \in \mathbb{R}^{V(G^\star)}$ that shows that $R_{\text{stab}}(G^\star)$ does not have the persistency property. Let A, b be as in the previous section, and denote by $a := A_{1,\star}$ the first row of A. We will define $c^\star$ via

$$c^\star_1 := \varepsilon \quad \text{and} \quad c^\star_v := a_j \cdot c^j_v \text{ for all } v \in V(G^j),\ j \in \{2, 3, \ldots, n\},$$

where $\varepsilon > 0$ is a positive constant that we will define later. Our first claim is independent of the specific choice of ε.



(a) Construction of $G^\star$ for Example 1.

(b) Construction of $G^\star$ for Example 2.

Figure 2: Construction of graph $G^\star$ for the two running examples as a 1-sum of G^{in} and copies of G^{out}. Pairs of nodes that are identified in the 1-sums are connected by dotted lines.

Claim 11. *Every $c^\star$-maximal stable set in $G^\star$ contains node $1 \in V(G^{in})$.*

Proof. By Claim 9 there exists, for each $j \in \{2, 3, \ldots, n\}$, a c^j-maximal stable set $S^j \subseteq V(G^j)$ that does not use v^j. Thus, the maximum objective value obtained on $V(G^\star \setminus \{1\})$ is $\sum_{j=2}^n a_j c^j(S^j)$, which is equal to the maximum objective value for all stable sets that do not contain node 1. Since $v^j \notin S^j$ for each j, the set $S^\star := \bigcup_{j=2}^n S^j \cup \{1\}$ is a stable set in $G^\star$ with objective value $\varepsilon + \sum_{j=2}^n a_j c^j(S^j) > \sum_{j=2}^n a_j c^j(S^j)$, which proves the claim. $\square$

Again, we verify Claim 11 for our two running examples.

Example 1 (Continued). For $R^{\text{oc5}}_{\text{stab}}$, $Ax \leq b$ consists only of the odd-cycle inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 1$, we have $a_j = 1$ for $j \in \{2, 3, 4, 5\}$. Hence, the objective vector is defined via $c^\star_1 = \varepsilon$, $c^\star_v = 1$ for all other nodes $v \in V(G^\star) \setminus \{1\}$, where a specific value of ε still has to be defined. Each $c^\star$-maximal stable set in $G^\star$ must contain node 1 since otherwise it would contain nodes A^2 or A^3, which we could replace by B^2 or B^3 without a decrease of the objective value. This in turn allows to include node 1 as well.

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, $Ax \leq b$ consists only of the triangle inequality $x_1 + x_2 + x_3 \leq 1$, we have $a_j = 1$ for $j \in \{2, 3\}$. Hence, the objective vector is defined via $c^\star_1 = \varepsilon$, $c^\star_j = \frac{1}{3}$, $c^\star_{A^j} = 1 + \frac{2}{3}$ and $c^\star_{B^j} = c^\star_{C^j} = c^\star_{D^j} = 1$ for $j \in \{2, 3\}$, where a specific value of ε still has to be defined. It is easy to see that $\{1, A^2, A^3\}$ is the unique $c^\star$-maximal stable set in $G^\star$.

To see that $R_{\text{stab}}(G^\star)$ does not have the persistency property, it suffices to establish the following claim, which then yields Theorem 4.

Claim 12. *For $\varepsilon > 0$ small enough, every $c^\star$-optimal point $x^\star \in R_{stab}(G^\star)$ satisfies $x^\star_1 = 0$.*

Let $x^\star$ be any $c^\star$-optimal point in $R_{\text{stab}}(G^\star)$. Observe that for each $j \in \{2, \ldots, n\}$ the sum of the $c^\star$-weights on the nodes in $V(G^j)$ only depends on the value of $x^\star_{v^j}$. In order to understand these contributions in terms of $x^\star_{v^j}$, let us introduce the function $f : [0, 1] \to \mathbb{R}$ defined via

$$f(y) := \max\left\{ {c^j}^\intercal x \mid x \in R_{\text{stab}}(G^j) \text{ and } x_{v^j} = y \right\},$$

Note that the definition is independent of j since all (G^j, c^j, v^j) are identical up to indexing. We observe that the restriction of $x^\star$ onto the coordinates corresponding to $V(G^{\text{in}})$ is an optimal solution for

$$\max\left\{ c'(x) \mid x \in R_{\text{stab}}(G^{\text{in}}) \right\} = \max\left\{ c'(x) \mid x \in R^{\text{edge}}_{\text{stab}}(G^{\text{in}}),\ Ax \leq b \right\}, \tag{1}$$

where $c'(x) := \varepsilon x_1 + \sum_{j=2}^n a_j f(x_j)$. Thus, we see that Claim 12 immediately follows from the following result.

Claim 13. *For $\varepsilon > 0$ small enough, every c'-optimal point $x \in R_{stab}(G^{in})$ satisfies $x_1 = 0$.*

Example 1 (Continued). For $R^{\text{oc5}}_{\text{stab}}$, we have

$$f(x_j) = \max\left\{x_{A^j} + x_{B^j} + x_{C^j} \mid x \in R^{\text{oc5}}_{\text{stab}}(K^j_3) \text{ and } x_{A^j} = x_j\right\}$$

for $j = 2, 3, 4, 5$, and due to $R^{\text{oc5}}_{\text{stab}}(K^j_3) = R^{\text{edge}}_{\text{stab}}(K^j_3)$ it follows that f attains its unique maximum at $x_j = \frac{1}{2}$. Consequently, any vector $x^{\text{LP}} \in R_{\text{stab}}(G^\star)$ with $x^{\text{LP}}_v = \frac{1}{2}$ for all $v \in V(G^\star) \setminus \{1\}$ is $c^\star$-maximal if we ignore the objective contribution of $\varepsilon \cdot x_1$ (see Figure 4a for an illustration). Now, setting $x_2 = x_3 = x_4 = x_5 = \frac{1}{2}$ leaves no slack in the odd-cycle inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$. Hence, a positive x_1-variable would require a reduction of x_j for some $j \in \{2, 3, 4, 5\}$, which in turn reduced $f(x_j)$. Hence, for sufficiently small $\varepsilon > 0$, such a reduction is not profitable, which proves Claim 13 for this example. Figure 4a depicts $G^\star$, $c^\star$, a $c^\star$-optimal point $x^\star \in R_{\text{stab}}(G^\star)$ and a $c^\star$-maximal stable set for $\varepsilon = \frac{1}{20}$.

Actually, the fact that there is no slack in the odd cycle inequality to set $x_1 > 0$ in Example 1 is not a coincidence, it follows from the following result by Sewell on the defect of facets of the stable-set polytope.

Proposition 14 (Corollary 3.4.3 in [13])**.** *Let $\sum_{j=1}^n a_j x_j \leq b_1$ be a facet-defining inequality for the stable set polytope of a graph on n nodes that is neither a bound nor an edge inequality. Then we have*

$$a_1 \leq \sum_{j=1}^n a_j - 2b_1.$$

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, we have

$$f(x_j) = \max\left\{\frac{1}{3}x_{A'^j} + \frac{5}{3}x_{A^j} + x_{B^j} + x_{C^j} + x_{D^j} \mid x \in R^{\text{oc}}_{\text{stab}}(G^j) \text{ and } x_{A'^j} = x_j\right\},$$

for $j = 2, 3$, where $G^j = K^j_4 \oplus^{A^j}_{A^j} K^j_2$, and for each j, f attains its unique maximum at $\hat{x}_v = \frac{1}{3}$ for $v \in \{A^j, B^j, C^j, D^j\}$ and $\hat{x}_{A'^j} = \frac{2}{3}$. However, in this case, setting $x_j = \frac{2}{3}$ for $j = 2, 3$ results in an infeasible solution of optimization problem (1), hence Claim 13 does not follow as easily as for Example 1.

As illustrated by Example 2, the general proof of Claim 13 is a bit more technical than for Example 1 since we have to ensure that all inequalities $Ax \leq b$ and all edge inequalities are satisfied, which is not always the case for the optimal solutions obtained when considering $f(x_j)$ separately for each j. To overcome this difficulty for the first inequality $a^\intercal x \leq b_1$ of the system $Ax \leq b$, it will be convenient to consider the function $g : [0, \infty] \to \mathbb{R}$ defined via

$$g(z) := \max\left\{\sum_{j=2}^n a_j f(x_j) \mid a^\intercal x \leq z,\ x \in R^{\text{edge}}_{\text{stab}}(G^{\text{in}})\right\}.$$

The intuition behind the proof of Claim 13 is the following: First, note that $c'(x)$ is the sum of εx_1 and the objective function defining g. Function $g(z)$ represents the contribution to the objective value of G^j for $j = 2, \ldots, n$ as a function of the right-hand-side of the inequality $a^\intercal x \leq z$, and reaches its maximum precisely at the value b_1, which defines the first "missing facet", that is, the first inequality in $Ax \leq b$. Hence, the contribution of the graphs G^j for $j = 2, \ldots, n$ will be maximized when the missing inequality $a^\intercal x \leq b_1$ is satisfied. Moreover, by Proposition 14, we know that there is no slack, meaning that any feasible solution with $x_1 > 0$ comes at the cost of decreasing the value of one of the x_j variables, which is not profitable if ε is small enough. The formalization of the fact that increasing x_1 is not profitable is the following claim.

Claim 15. *The functions f and g are concave. Moreover, g is strictly monotonically increasing on $[0, b_1]$.*

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, we have

$$g(z) := \max \left\{ \sum_{j \in \{2,3\}} f(x_j) \mid x_1 + x_2 + x_3 \leq z,\ x \in R^{\text{edge}}_{\text{stab}}(C_3) \right\}.$$

Function g is illustrated in Figure 3. It is clearly concave, and linear and strictly monotonically increasing on $[0, b_1] = [0, 1]$, hence Claim 15 is satisfied.

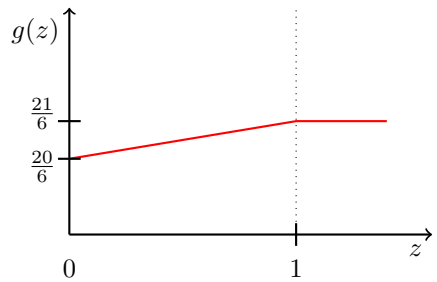


Figure 3: Illustration of function $g(z)$ for Example 2.

Proof of Claim 13. Letting

$$\gamma := \min \{x_1 \mid x \text{ vertex of } R_{\text{stab}}(G^\star) \text{ with } x_1 > 0\} \in (0, 1], \text{ and}$$
$$\lambda := \min \{\gamma/(A_{i,1} + \cdots + A_{i,n}) \mid i \in \{1, 2, \ldots, m\}\} \in (0, 1),$$

we claim that every choice of ε with

$$0 < \varepsilon < \lambda(g(b_1) - g(b_1 - a_1\gamma))$$

satisfies the assertion. First, we need to verify that the right-hand side is positive. To this end, note that $a_1 \leq b_1$ and hence $0 \leq b_1 - a_1\gamma < b_1$. So, by Claim 15 we have

$$g(b_1 - a_1\gamma) < g(b_1), \tag{2}$$

which yields positivity of the right-hand side.

Next, let ε be as above. For the sake of contradiction, assume that there exists a c'-optimal solution $x^\star \in R_{\text{stab}}(G^{\text{in}})$ with $x^\star_1 > 0$. Note that $x^\star$ can be extended to a $c^\star$-optimal solution over $R_{\text{stab}}(G^\star)$, which we may assume to be a vertex of $R_{\text{stab}}(G^\star)$, and hence $x^\star_1 \geq \gamma$. Let $\hat{x}^0 \in R_{\text{stab}}(G^{\text{in}})$ be equal to $x^\star$, except for $\hat{x}^0_1 := 0$. Moreover, let $\hat{x}^1 \in \mathbb{R}^{V(G^{\text{in}})}$ be a maximizer of $g(b_1)$, which may not be contained in $R_{\text{stab}}(G^{\text{in}})$. Now consider the vector $\hat{x}^\lambda := (1 - \lambda)\hat{x}^0 + \lambda\hat{x}^1$. To obtain the desired contradiction, we will show that $\hat{x}^\lambda$ is contained in $R_{\text{stab}}(G^{\text{in}})$ and that $c'(\hat{x}^\lambda) > c'(x^\star)$.

Since $\hat{x}^0$ and $\hat{x}^1$ both lie in $R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$, also x^λ lies in $R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$. Let $i \in \{1, 2, \ldots, m\}$. By Claim 10, $A_{i,1} \geq 1$ holds, which implies $A_{i,\star}\hat{x}^0 \leq A_{i,\star}x^\star - \gamma \leq b_i - \gamma$. We obtain

$$A_{i,\star}\hat{x}^\lambda = A_{i,\star}\hat{x}^0 + \lambda A_{i,\star}(\hat{x}^1 - \hat{x}^0) \leq b_i - \gamma + \lambda(A_{i,1} + \cdots + A_{i,n}) \leq b_i,$$

where the second inequality follows from the fact that each coordinate of $\hat{x}^1 - \hat{x}^0$ is bounded by 1, and the last inequality holds by the definition of λ. This shows that $\hat{x}^\lambda$ is contained in $R_{\text{stab}}(G^{\text{in}})$.

For the objective value of $\hat{x}^1$ we clearly have $c'(\hat{x}^1) \geq g(b_1)$. Moreover, since $\hat{x}^0_1 = 0$ we have

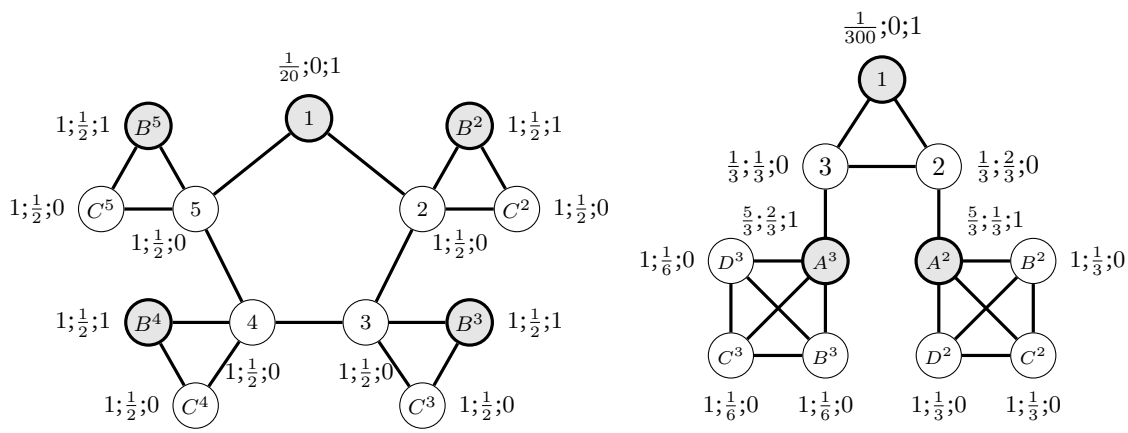
$$c'(\hat{x}^0) \leq g(a^\intercal \hat{x}^0) \leq g(b_1 - a_1\gamma) < g(b_1),$$

where the latter two inequalities again follow from Claim 15 and (2). Observe that concavity of f and nonnegativity of a imply concavity of $c'(x)$, which yields $c'(\hat{x}^\lambda) \geq (1-\lambda)c'(\hat{x}^0) + \lambda c'(\hat{x}^1)$. We obtain

$$\begin{aligned} c'(x^\star) - c'(\hat{x}^\lambda) &\leq \left(\varepsilon + c'(\hat{x}^0)\right) - \left(c'(\hat{x}^0) - \lambda(c'(\hat{x}^0) - c'(\hat{x}^1))\right) = \varepsilon + \lambda(c'(\hat{x}^0) - c'(\hat{x}^1)) \\ &\leq \varepsilon + \lambda(g(b_1 - a_1\gamma) - g(b_1)) < 0, \end{aligned}$$

where the last inequality holds by definition of ε and due to (2). □

Example 2 (Continued). For $R^{\text{oc}}_{\text{stab}}$, one can check (for example with a computer program) that $\gamma = \frac{1}{3}$ and $\lambda = \frac{1}{9}$. For every $0 < \varepsilon < \frac{1}{9}\left(g(1) - g(1-\frac{1}{3})\right) = \frac{1}{9}\left(\frac{7}{2} - \frac{31}{9}\right) = \frac{1}{162}$, Claim 13 is satisfied. Figure 4b depicts $G^\star$, $c^\star$, a $c^\star$-optimal point $x^\star \in R_{\text{stab}}(G^\star)$ and a $c^\star$-maximal stable set for $\varepsilon = \frac{1}{300}$.



(a) $G^\star$, $c^\star$ and LP/IP maxima for Example 1.

(b) $G^\star$, $c^\star$ and LP/IP maxima for Example 2.

Figure 4: Illustration of $G^\star$ for our running examples. For each node $v \in V(G^\star)$, the triple $c^\star_v; x^{\text{LP}}_v; x^{\text{IP}}_v$ denotes the corresponding coefficient in $c^\star$, the value of component v in a $c^\star$-maximal solution over $R_{\text{stab}}(G^\star)$ and the value in the unique $c^\star$-maximal stable set, respectively.

To conclude the proof of Theorem 4, it remains to prove Claim 15. The fact that f and g are concave is a simple consequence of the next basic lemma.

Lemma 16. *Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope, let $c, a \in \mathbb{R}^n$ and let $\ell := \min\{a^\intercal x \mid x \in P\}$. The functions $h^=, h^\leq : [\ell, \infty) \to \mathbb{R}$ defined via $h^=(\beta) = \max\{c^\intercal x \mid x \in P, a^\intercal x = \beta\}$ and $h^\leq(\beta) = \max\{c^\intercal x \mid x \in P, a^\intercal x \leq \beta\}$ are concave. Moreover, there exists a number $\beta^\star \in [\ell, \infty)$ such that $h^=$ and $h^\leq$ are identical and strictly monotocially increasing on the interval $[\ell, \beta^\star]$, and $h^\leq$ is constant on the interval $[\beta^\star, \infty)$.*

Proof. See Appendix A. □

Proof of Claim 15. From Lemma 16 it is clear that f is concave. By rewriting

$$g(z) = \max\left\{\sum_{j=2}^{n} a_j \cdot \sum_{v \in V(G^j)} c^j_v x_v \mid \sum_{j=1}^{n} a_j x_j \leq z,\ x \in R^{\text{edge}}_{\text{stab}}(G^{\text{in}}) \oplus^2_{v^2} R_{\text{stab}}(G^2) \oplus^3_{v^3} \cdots \oplus^n_{v^n} R_{\text{stab}}(G^n)\right\},$$

we also see that g is concave. Moreover, again by Lemma 16, there exists some $\beta^\star \geq 0$ such that g is strictly monotonically increasing on the interval $[0, \beta^\star]$, and constant on $[\beta^\star, \infty)$. It suffices to show that $\beta^\star \geq b_1$. To this end, let us get back to our initial definition of g, and let $\hat{x} \in R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$ be a maximizer for $g(\infty)$. Note that $\beta^\star \geq a^\intercal \hat{x}$ by definition of $\beta^\star$, and hence we have to show that $\hat{x}$ satisfies $a^\intercal \hat{x} \geq b_1$.

Since the objective value of $\hat{x}$ does not depend on $\hat{x}_1$, we may assume that $\hat{x}_1 = 0$. By the construction of G^j and c^j, we know that f attains its unique maximum at $y^\star \geq \frac{1}{2}$. This implies $0 \leq \hat{x}_j \leq y^\star$ for $j = 2, 3, \ldots, n$. Moreover, we claim that also $\hat{x}_j \geq 1 - y^\star$ holds. Suppose not, then none of the edge inequalities involving x_j is tight. Then $\hat{x}_j < 1 - y^\star \leq y^\star$ shows that increasing $\hat{x}_j$ would improve the objective value, which in turn contradicts optimality of $\hat{x}$. Consequently, even $1 - y^\star \leq \hat{x}_j \leq y^\star$ holds for $j = 2, 3, \ldots, n$.

Let $J(\alpha) \coloneqq \{2 \leq j \leq n \mid \hat{x}_j = \alpha\}$ for $\alpha \in [1 - y^\star, y^\star]$. We will show that $a(J(\alpha)) \geq a(J(1-\alpha))$ holds for all $\alpha \in (1/2, y^\star]$, where $a(J(\alpha))$ shall denote $\sum_{j \in J(\alpha)} a_j$. Note that this implies the claim since for each $\alpha \in (1/2, y^\star]$ we then have

$$\begin{aligned}
\sum_{j \in J(\alpha)} a_j \hat{x}_j + \sum_{j \in J(1-\alpha)} a_j \hat{x}_j &= \sum_{j \in J(\alpha)} a_j \alpha + \sum_{j \in J(1-\alpha)} a_j (1-\alpha) \\
&= \alpha \cdot \underbrace{[a(J(\alpha)) - a(J(1-\alpha))]}_{\geq 0} + a(J(1-\alpha)) \\
&\geq \tfrac{1}{2} \cdot [a(J(\alpha)) - a(J(1-\alpha))] + a(J(1-\alpha)) \\
&= \sum_{j \in J(\alpha)} a_j \tfrac{1}{2} + \sum_{j \in J(1-\alpha)} a_j \tfrac{1}{2}
\end{aligned}$$

and hence

$$\begin{aligned}
a^\intercal \hat{x} = \sum_{j=2}^{n} a_j \hat{x}_j &= \sum_{j \in J(1/2)} a_j \hat{x}_j + \sum_{\alpha \in (1/2, y^\star]} \Big(\sum_{j \in J(\alpha)} a_j \hat{x}_j + \sum_{j \in J(1-\alpha)} a_j \hat{x}_j \Big) \\
&\geq \sum_{j \in J(1/2)} a_j \tfrac{1}{2} + \sum_{\alpha \in (1/2, y^\star]} \Big(\sum_{j \in J(\alpha)} a_j \tfrac{1}{2} + \sum_{j \in J(1-\alpha)} a_j \tfrac{1}{2} \Big) = \tfrac{1}{2} \sum_{j=2}^{n} a_j \geq b_1,
\end{aligned}$$

where the last inequality follows from Proposition 14.

For the sake of contradiction, assume that $a(J(\alpha)) < a(J(1-\alpha))$ holds for some $\alpha \in (1/2, y^\star]$. For a sufficiently small $\varepsilon' > 0$, the solution $\hat{x}' \in \mathbb{R}^{V(G^{\text{in}})}$ defined via

$$\hat{x}'_j \coloneqq \begin{cases} \hat{x}_j + \varepsilon' & \text{if } j \in J(1-\alpha) \\ \hat{x}_j - \varepsilon' & \text{if } j \in J(\alpha) \\ \hat{x}_j \text{ otherwise} & \end{cases} \qquad \text{for } j = 1, 2, \ldots, n$$

is still contained in $R^{\text{edge}}_{\text{stab}}(G^{\text{in}})$. To see this, observe that $\hat{x}'_j \geq 0$ holds for all $j \in V(G^{\text{in}})$ since we only decrease entries that are at least $1/2$. Moreover, edge inequalities that are tight for $\hat{x}$ remain tight for $\hat{x}'$, since either none or both of its two node values are modified, where in the latter case, the value is increased by ε' for one node and decreased by ε' for the other. Finally, edge inequalities that are not tight for $\hat{x}$ will not be violated if we choose ε' sufficiently small. For the objective values we obtain

$$\begin{aligned}
\sum_{j=2}^{n} a_j (f(\hat{x}'_j) - f(\hat{x}_j)) &= \sum_{j \in J(1-\alpha)} a_j (f(\hat{x}'_j) - f(\hat{x}_j)) + \sum_{j \in J(\alpha)} a_j (f(\hat{x}'_j) - f(\hat{x}_j)) \\
&= a(J(1-\alpha)) \cdot \big(f(1 - \alpha + \varepsilon') - f(1-\alpha) \big) + a(J(\alpha)) \cdot \big(f(\alpha - \varepsilon') - f(\alpha) \big).
\end{aligned}$$

We also assume that ε' is small enough to guarantee $1 - \alpha + \varepsilon' < \alpha - \varepsilon'$. Since f is concave and monotonically increasing in $[0, y^\star]$, we obtain $f(1 - \alpha + \varepsilon') - f(1-\alpha) \geq f(\alpha) - f(\alpha - \varepsilon')$. Together with the assumption $a(J(1-\alpha)) > a(J(\alpha))$, this shows that the objective value of $\hat{x}'$ is strictly larger than that of $\hat{x}$, a contradiction to the optimality of $\hat{x}$ (see Figure 5 for an illustration). $\square$

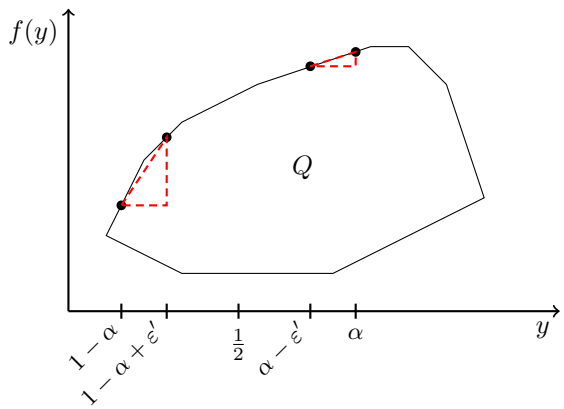


Figure 5: Illustration of modifications in the proof of Claim 15.

5 Concluding remarks

We have shown that persistency is an exceptional property for linear programming stable-set relaxations. Apart from studying nonlinear relaxations (such as those stemming from semidefinite relaxations), it is natural to ask whether this is also the case for other polytopes for which persistency was established.

The most interesting candidate is certainly the unconstrained quadratic binary programming problem, which is equivalent to the maximum cut problem. The standard *McCormick relaxation* also has the (weak and strong) persistency property [5]. In fact, there is a strong relationship to the stable-set problem as both problems can be easily reduced to each other. Polyhedrally speaking, each polytope (relaxation or integer hull) can be obtained as a face of the polytope of the other problem, potentially after removing constraints that are redundant for a given objective vector [5]. Although this was used to show that the McCormick relaxation has the persistency property, the non-existence of the property for tighter relaxations does not carry over in a straight-forward manner. Thus, we leave the resolution of this question as an open problem.

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A Deferred proofs

We repeat the statements of Lemma 8 and Lemma 16 provide their proofs.

Lemma 8. *Let $P, Q \subseteq \mathbb{R}^n$ be polytopes. If there exists a vector $c \in \mathbb{R}^n$ such that $\dim(\mathrm{opt}(Q, c)) < \dim(\mathrm{opt}(P, c))$, then there exists a vector $c' \in \mathbb{R}^n$ such that $\mathrm{opt}(Q, c')$ is a vertex of Q, while $\mathrm{opt}(P, c')$ is not a vertex of P.*

Proof. Let $c' \in \mathbb{R}^n$ be such that $\dim(\mathrm{opt}(Q, c')) < \dim(\mathrm{opt}(P, c'))$ holds, and among those, such that $\dim(\mathrm{opt}(Q, c'))$ is minimum. Clearly, c' is well-defined since $c' \coloneqq c$ satisfies the conditions.

Assume, for the sake of contradiction, that $\dim(\mathrm{opt}(Q, c')) > 0$. Let $F \coloneqq \mathrm{opt}(P, c')$ and $G \coloneqq \mathrm{opt}(Q, c')$. Let $F_1, F_2, \ldots, F_k$ be the facets of F. By $n(F, F_i)$ we denote the set of vectors $w \in \mathbb{R}^n$ such that $\mathrm{opt}(F, w) \supseteq F_i$. Since F is a polytope, $\bigcup_{i \in \{1,2,\ldots,k\}} n(F, F_i)$ contains a basis U of $\mathbb{R}^n$. Moreover, not all vectors $u \in U$ can lie in $\mathrm{aff}(G)^\perp$, the orthogonal complement of $\mathrm{aff}(G)$, since then $\mathrm{aff}(G)^\perp = \mathbb{R}^n$ would hold, contradicting $\dim(G) > 0$. Let $u \in U \setminus \mathrm{aff}(G)^\perp$.

Now, for a sufficiently small $\varepsilon > 0$, $\mathrm{opt}(P, c' + \varepsilon u) \supseteq F_i$ for some $i \in \{1, 2, \ldots, k\}$, and $\mathrm{opt}(Q, c' + \varepsilon u)$ is a proper face of G. Thus, $c' + \varepsilon u$ satisfies the requirements at the beginning of the proof. However, $\dim(\mathrm{opt}(Q, c' + \varepsilon u)) < \dim(G)$ contradicts the minimality assumption, which concludes the proof. □

Lemma 16. *Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope, let $c, a \in \mathbb{R}^n$ and let $\ell \coloneqq \min\{a^\intercal x \mid x \in P\}$. The functions $h^=, h^\leq : [\ell, \infty) \to \mathbb{R}$ defined via $h^=(\beta) = \max\{c^\intercal x \mid x \in P, a^\intercal x = \beta\}$ and $h^\leq(\beta) = \max\{c^\intercal x \mid x \in P, a^\intercal x \leq \beta\}$ are concave. Moreover, there exists a number $\beta^\star \in [\ell, \infty)$ such that $h^=$ and $h^\leq$ are identical and strictly monotocially increasing on the interval $[\ell, \beta^\star]$, and $h^\leq$ is constant on the interval $[\beta^\star, \infty)$.*

Proof. Let $Q \coloneqq \{\binom{y_1}{y_2} \mid \exists x \in P : a^\intercal x = y_1,\ c^\intercal x = y_2\} \subseteq \mathbb{R}^2$ be the projection of P along a and c. By construction, $h^\leq(\beta) = \max\{y_2 \mid y \in Q,\ y_1 \leq \beta\}$ holds. Considering that Q is a polytope of dimension at most 2, the claimed properties of $h^\leq$ and $h^=$ are obvious (see Figure 6). □

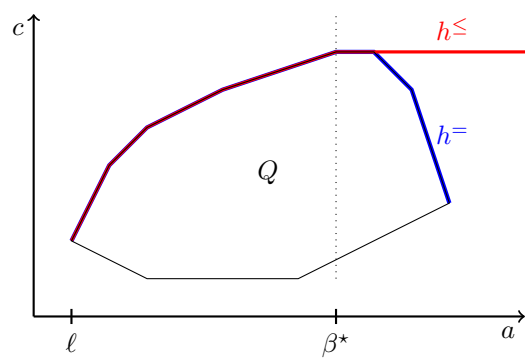

Figure 6: Illustration of Lemma 16. The graph of $h^{\leq}$ is highlighted in red, while that of $h^{=}$ is highlighted in blue.